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Neugebauer–Kramer solutions of the Ernst equation in Hirota's direct method

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Abstract. We prove analytically the Sasa–Satsuma conjecture which states that their solution of bilinear form of the Ernst equation gives the Neugebauere–Kramer solution in particular cases. This proof relates Hirota's direct method with the Bäcklund transformation method and opens the way towards the comprehensive interpretation of the Ernst equation.

1. Introduction

In [1, 2] one of the authors (TF) discussed the stationary axisymmetric solution, Tomimatsu–Sato solution [3] (hereafter TS solution) of the Einstein equation in the framework of Hirota's direct method (direct method) [4]. This paper describes further development of those papers.

The motivation to study the solutions of the Einstein's equation in the direct method is as follows. In the direct method integrability is reduced to Pfaffian's identities and N-soliton solutions can be constructed in very simple and systematic ways. This is a great merit of the direct method. However, this is the case in typical integrable systems such as the KdV, KP and Toda lattice equations etc, which can be described in the 'complete' bilinear forms. Here 'complete' means that these equations are expressed in terms of the Hirota's derivatives and constant coefficients. The Ernst equation is indeed beyond this category. The TS solutions, its special series of solutions, satisfy some bilinear forms. However, the bilinear forms are not complete in the sense that they involve ordinary derivatives and that the coefficients of differentiated terms are not constants but functions of independent variables. It also can not be expressed in the naive Lax pairs [5, 6]. Therefore, it was not clear whether the direct method works well in this case. In [1] it was proved that the direct method works well in the TS solutions. However, the proof (whose meaning will be described in the last section) was complete in the restricted case of one dimension, the Weyl solution, and incomplete in full two dimensions, the TS solution.

As is well known, there are many approaches to integrable systems; inverse scattering, Bäcklund transformation, Lax representation, Sato theory etc. The direct method is one among them and they are closely connected with one another. So it is natural to consider that the difficulties mentioned above may be overcome by adopting the various techniques other than the direct method. This paper is one of such trials.

Using the Bäcklund transformation, Y Nakamura [7] found two series of solutions to the Ernst equation. Unfortunately these solutions are not physical. In the direct method, the Ernst equation is decomposed to the bilinear forms in many ways. By rewriting the

Bäcklund transformation adopted by Nakamura in bilinear forms, one of the authors (NS) and Satsuma succeeded to extend Nakamura's solutions (SS solutions) and showed that they include the physical soliton solutions, Neugebauer–Kramer solutions (NK solutions) [8] in particular cases. We call the latter statement the Sasa–Satsuma conjecture (SS conjecture) since Sasa and Satsuma showed that their solutions include $2N$ soliton solution of the NK solution for the cases of $N = 2$ and 3 by means of computer program 'REDUCE 3' [9]. Namely, the SS solution is linked with the NK solution through the SS conjecture. The NK solutions are reduced to the TS solutions in a limit. So the analytical proof of the SS conjecture opens the way towards the comprehensive understanding of the Ernst equation and complements the above-mentioned deficit of the previous works.

The central purpose of this work is to prove the SS conjecture analytically for general N. However, using a merit of the direct method, we can extend the SS solutions furthermore and obtain more general solutions.

This paper is organized as follows. In section 2, we review the SS conjecture. Section 3 only contains the analytical proof of the SS conjecture. Straightforward but tedious calculations in the proof are referred to in the appendices. The extension of the SS solutions is discussed in section 4. Section 5 is devoted to concluding remarks and discussion.

2. The Ernst equation and Sasa–Satsuma conjecture

The stationary axisymmetric vacuum gravitational field equations are reduced to the following two equations

$$\tilde{f}\left(\tilde{f}_{\rho\rho} + \frac{1}{\rho}\tilde{f}_\rho + \tilde{f}_{zz}\right) - \tilde{f}_\rho^2 - \tilde{f}_z^2 + \psi_\rho^2 + \psi_z^2 = 0 \tag{2.1}$$

$$\tilde{f}\left(\psi_{\rho\rho} + \frac{1}{\rho}\psi_\rho + \psi_{zz}\right) - 2\tilde{f}_\rho\psi_\rho - 2\tilde{f}_z\psi_z = 0 \tag{2.2}$$

where ρ and z are usual cylindrical coordinates and subscripts denote partial derivatives such as $\tilde{f}_{\rho\rho} \equiv \frac{\partial^2 \tilde{f}}{\partial \rho^2}$ etc. Defining a complex function ξ by

$$\xi \equiv \frac{1 - \tilde{f} - \mathrm{i}\psi}{1 + \tilde{f} + \mathrm{i}\psi} \tag{2.3}$$

we have the Ernst equation [10]:

$$(\xi\xi^* - 1)\left(\xi_{\rho\rho} + \frac{1}{\rho}\xi_\rho + \xi_{zz}\right) - 2\xi^*(\xi_\rho^2 + \xi_z^2) = 0. \tag{2.4}$$

By transforming the dependent variables as

$$\tilde{f} \equiv \frac{F}{G} \qquad \psi \equiv \frac{H}{G} \tag{2.5}$$

and introducing K by

$$K = \frac{H^2 + F^2}{G} \tag{2.6}$$

equations (2.1) and (2.2) are decomposed into

$$\left[D_\rho^2 + \frac{1}{\rho}D_\rho + D_z^2\right] G \cdot F = 0 \tag{2.7}$$

$$\left[D_\rho^2 + \frac{1}{\rho}D_\rho + D_z^2\right] H \cdot F = 0 \tag{2.8}$$

$$\left[D_\rho^2 + \frac{1}{\rho}D_\rho + D_z^2\right] K \cdot F = 0 \tag{2.9}$$

where D_ρ and D_z are Hirota's D-operators with respect to ρ and z. Using the Bäcklund transformation, Y Nakamura found a series of solutions to equations (2.1), (2.2) [7],

$$\tilde{f} = \frac{\rho^{n-1}A^{(n)}}{A^{(n-1)}} \qquad \psi = \frac{\mathrm{i}\rho^{n-1}\tilde{A}^{(n+1)}}{A^{(n-1)}} \tag{2.10}$$

or to equations (2.7)–(2.9)

$$F = \rho^{n_0}A^{(n)} \qquad G = \rho^{n_0-n+1}A^{(n-1)} \qquad K = \rho^{n_0+n-1}A^{(n+1)} \qquad H = \mathrm{i}\rho^{n_0}\tilde{A}^{(n+1)} \tag{2.11}$$

with $n_0 = n(n-2)/2$. Here the determinants $A^{(n)}$ and $\tilde{A}^{(n)}$ are given by

$$A^{(n)} = \begin{vmatrix} u_0 & iu_1 & i^2u_2 & \dots & i^{n-1}u_{n-1} \\ iu_1 & u_0 & iu_1 & \dots & i^{n-2}u_{n-2} \\ i^2u_2 & iu_1 & u_0 & \dots & i^{n-3}u_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ i^{n-1}u_{n-1} & i^{n-2}u_{n-2} & i^{n-3}u_{n-3} & \dots & u_0 \end{vmatrix} \tag{2.12}$$

and

$$\tilde{A}^{(n)} = A^{(n)}\begin{bmatrix} 1 \\ n \end{bmatrix} \tag{2.13}$$

where the minor $A^{(n)}\begin{bmatrix} i \\ j \end{bmatrix}$ is defined by deleting the ith row and the jth column from $A^{(n)}$. The elements of the determinants (2.12) and (2.13) satisfy the recurrence relations:

$$\left(\partial_\rho + \frac{m-1}{\rho}\right)u_m = -\partial_z u_{m-1}$$
$$\left(\partial_\rho - \frac{m}{\rho}\right)u_{m-1} = \partial_z u_m \qquad (m = 1, 2, 3, \dots). \tag{2.14}$$

It is noticed that the Jacobi identity reads

$$A^{(n+1)}A^{(n-1)} = [A^{(n)}]^2 - [\tilde{A}^{(n+1)}]^2. \tag{2.15}$$

By substituting equations (2.11) into equations (2.7)–(2.9), we find that $A^{(n)}$ and $\tilde{A}^{(n)}$ satisfy the bilinear forms,

$$\left[D_\rho^2 + \frac{1}{\rho}D_\rho + D_z^2\right](\rho^{n_0-n+1}A^{(n-1)}) \cdot (\rho^{n_0}A^{(n)}) = 0 \tag{2.16}$$

$$\left[D_\rho^2 + \frac{1}{\rho}D_\rho + D_z^2\right](\rho^{n_0+n-1}A^{(n+1)}) \cdot (\rho^{n_0}A^{(n)}) = 0 \tag{2.17}$$

$$\left[D_\rho^2 + \frac{1}{\rho}D_\rho + D_z^2\right](\rho^{n_0}\tilde{A}^{(n+1)}) \cdot (\rho^{n_0}A^{(n)}) = 0. \tag{2.18}$$

From these facts Sasa and Satsuma found a new series of exact solutions (SS solutions) [9]:

$$\tilde{f} = \frac{A^{(n)}}{\tilde{A}^{(n+1)} - \frac{1}{4a_n}\rho^{n-1}A^{(n+1)} + a_n\rho^{1-n}A^{(n-1)}} \tag{2.19}$$

$$\psi = \frac{\mathrm{i}[\frac{1}{4a_n}\rho^{n-1}A^{(n+1)} + a_n\rho^{1-n}A^{(n-1)}]}{\tilde{A}^{(n+1)} - \frac{1}{4a_n}\rho^{n-1}A^{(n+1)} + a_n\rho^{1-n}A^{(n-1)}} \tag{2.20}$$

with a constant a_n. Furthermore, it has been argued that the SS solutions (2.19) and (2.20) include the NK solutions in the particular choice of an initial value u_0 and constant a_{2N} as equations (2.30) and (2.33), respectively (SS conjecture). They checked this conjecture numerically for the $N = 1, 2$ and $N = 3$ cases. The main purpose of this paper is to prove analytically the SS conjecture for arbitrary N. Let us explain the SS conjecture in more detail. Let us formulate this conjecture for arbitrary N.

The NK solutions are given by [8]

$$\xi_{2N} = \frac{g_{2N}}{f_{2N}} \tag{2.21}$$

where f_{2N} and g_{2N} are the double Casorati determinants:

$$f_{2N} = \begin{vmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{N-1} & z_2^{N-1} & \dots & z_{2N}^{N-1} \\ S_1 & S_2 & \dots & S_{2N} \\ z_1 S_1 & z_2 S_2 & \dots & z_{2N} S_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{N-1} S_1 & z_2^{N-1} S_2 & \dots & z_{2N}^{N-1} S_{2N} \end{vmatrix} \tag{2.22}$$

$$g_{2N} = \begin{vmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{N} & z_2^{N} & \dots & z_{2N}^{N} \\ S_1 & S_2 & \dots & S_{2N} \\ z_1 S_1 & z_2 S_2 & \dots & z_{2N} S_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{N-2} S_1 & z_2^{N-2} S_2 & \dots & z_{2N}^{N-2} S_{2N} \end{vmatrix} \tag{2.23}$$

and

$$S_j = r_j \mathrm{e}^{\mathrm{i}\omega_j} \qquad r_j = \sqrt{\rho^2 + (z + z_j)^2} \qquad (j = 1, 2, \dots, 2N) \tag{2.24}$$

with real parameters z_j and ω_j. From the definition of $\tilde{f}$ and ψ it follows that

$$\tilde{f} = \frac{f_{2N} f_{2N}^* - g_{2N} g_{2N}^*}{(f_{2N} + g_{2N})(f_{2N}^* + g_{2N}^*)} \qquad \psi = \frac{\mathrm{i}(f_{2N}^* g_{2N} - f_{2N} g_{2N}^*)}{(f_{2N} + g_{2N})(f_{2N}^* + g_{2N}^*)}. \tag{2.25}$$

The SS conjecture states that the SS solutions (2.19) and (2.20) with particular conditions (2.30) and (2.33) satisfy equation (2.25) when the NK solutions (2.22), (2.23) have been substituted into the right-handside of equation (2.25). Given the NK solution f_{2N}, g_{2N}, if the determinants $A^{(n)}$ and $\tilde{A}^{(n)}$ take the forms,

$$\rho^{-2N+1} A^{(2N-1)} = \frac{-B}{a_{2N}} g_{2N} f_{2N}^* \tag{2.26}$$

$$A^{(2N)} + \tilde{A}^{(2N+1)} = -2B f_{2N} f_{2N}^* \tag{2.27}$$

$$A^{(2N)} - \tilde{A}^{(2N+1)} = 2B g_{2N} g_{2N}^* \tag{2.28}$$

$$\rho^{2N-1} A^{(2N+1)} = 4a_{2N} B g_{2N}^* f_{2N} \tag{2.29}$$

equation (2.25) is automatically satisfied. Here B is a common factor to be determined. Sasa and Satsuma found for $N = 1$, 2 and 3 that equations (2.26)–(2.29) are satisfied when u_0 and a_{2N} are chosen as follows

$$u_0 = b_{2N} \sum_{j=1}^{2N} C_j \frac{\rho}{S_j^*} \tag{2.30}$$

with

$$C_j \equiv (-1)^{j+1} Z_{2N} \begin{bmatrix} 2N \\ 1 \end{bmatrix} \qquad \text{and} \qquad b_{2N} = (-1)^{N-1} (\tfrac{1}{4})^{N(N-1)} Z_{2N}^{-2(N-1)} \tag{2.31}$$

where Z_{2N} is Vandermonde's determinant given by

$$Z_{2N} = \begin{vmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_{2N} \\ z_1^2 & z_2^2 & \dots & z_{2N}^2 \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{2N-1} & z_2^{2N-1} & \dots & z_{2N}^{2N-1} \end{vmatrix}. \tag{2.32}$$

The constant a_{2N} is

$$a_{2N} = -4^{N-1} Z_{2N} b_{2N}. \tag{2.33}$$

In this case B becomes

$$B = (\rho^{N_0} S_1^* S_2^* \dots S_{2N}^*)^{-1}. \tag{2.34}$$

Then the analytical proof of the SS conjecture for an arbitrary N is reduced to the proof of a set of equations (2.26)–(2.29).

3. The proof of the Sasa–Satsuma conjecture

In this section we give the analytical proof of the SS conjecture. From equations (2.26)–(2.29) it is estimated that $A^{(n)}$ and $\tilde{A}^{(n)}$ should be factorized as

$$A^{(2N-1)} = \alpha_N \beta_N \tag{3.1}$$

$$A^{(2N)} + \tilde{A}^{(2N+1)} = \beta_N \gamma_N \tag{3.2}$$

$$A^{(2N)} - \tilde{A}^{(2N+1)} = \alpha_N \delta_N \tag{3.3}$$

$$A^{(2N+1)} = \gamma_N \delta_N \tag{3.4}$$

where the determinants $\alpha_N, \beta_N, \gamma_N$ and δ_N are defined as

$$\alpha_N = \gamma_N \begin{bmatrix} N \\ N \end{bmatrix} \tag{3.5}$$

$$\beta_N = \delta_N \begin{bmatrix} N+1 \\ N+1 \end{bmatrix} \tag{3.6}$$

$$\gamma_N = \begin{vmatrix} u_0 + u_2 & -u_1 - u_3 & u_2 + u_4 & \dots & (-1)^{N+1}(u_{N-1} + u_{N+1}) \\ u_1 + u_3 & u_0 - u_4 & -u_1 + u_5 & \dots & (-1)^N (u_{N-2} - u_{N+2}) \\ u_2 + u_4 & u_1 - u_5 & u_0 + u_6 & \dots & (-1)^{N-1}(u_{N-3} + u_{N+3}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{N-1} + u_{N+1} & u_{N-2} - u_{N+2} & u_{N-3} + u_{N+3} & \dots & u_0 + (-1)^{N+1} u_{2N} \end{vmatrix} \tag{3.7}$$

$$\delta_N = \begin{vmatrix} u_0 & -u_1 & u_2 & -u_3 & \dots & (-1)^N u_N \\ 2u_1 & u_0 - u_2 & -u_1 + u_3 & u_2 - u_4 & \dots & (-1)^{N-1}(u_{N-1} - u_{N+1}) \\ 2u_2 & u_1 - u_3 & u_0 + u_4 & -u_1 - u_5 & \dots & (-1)^{N-2}(u_{N-2} + u_{N+2}) \\ 2u_3 & u_2 - u_4 & u_1 + u_5 & u_0 - u_6 & \dots & (-1)^{N-3}(u_{N-3} - u_{N+3}) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2u_N & u_{N-1} - u_{N+1} & u_{N-2} + u_{N+2} & u_{N-3} - u_{N+3} & \dots & u_0 + (-1)^N u_{2N} \end{vmatrix}. \tag{3.8}$$

u_n can be constructed from the recursion relation (2.14) with the initial condition (2.30):

$$u_0 = x_0 \tag{3.9}$$

$$u_{2k+2} = \sum_{n=0}^{k} (-1)^{k+1-n} \frac{k+1}{k+1-n} {}_{k+n}C_{k-n} 2^{2n} x_n + 2^{2k+1} x_{k+1} \tag{3.10}$$

$$u_{2k+1} = \sum_{n=0}^{k} (-1)^{k+1-n} {}_{k+n}C_{k-n} 2^{2n} y_n \qquad (k = 0, 1, 2, \dots) \tag{3.11}$$

where

$$x_n = \frac{1}{\rho^{2n-1}} \sum_{j=1}^{2N} C_j e^{i\omega_j} r_j^{2n-1} \tag{3.12}$$

$$y_n = \frac{1}{\rho^{2n}} \sum_{j=1}^{2N} C_j e^{i\omega_j} (z + z_j) r_j^{2n-1} \qquad (n = 0, 1, 2, \ldots). \tag{3.13}$$

Substituting the special solutions (3.9)–(3.11) into equations (3.5)–(3.8) we obtain after tedious calculations (see appendix A)

$$\alpha_{2k} = (-1)^{k-1} 2^{(2k-1)^2} \begin{vmatrix} x_1 & y_1 & \ldots & x_{k-1} & y_{k-1} & x_k \\ y_1 & x_2 - x_1 & \ldots & y_{k-1} & x_k - x_{k-1} & y_k \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ x_{k-1} & y_{k-1} & \ldots & x_{2k-3} & y_{2k-3} & x_{2k-2} \\ y_{k-1} & x_k - x_{k-1} & \ldots & y_{2k-3} & x_{2k-2} - x_{2k-3} & y_{2k-2} \\ x_k & y_k & \ldots & x_{2k-2} & y_{2k-2} & x_{2k-1} \end{vmatrix} \tag{3.14}$$

$$\alpha_{2k-1} = (-1)^{k-1} 2^{(2k-2)^2} \begin{vmatrix} x_1 & y_1 & \ldots & x_{k-1} & y_{k-1} \\ y_1 & x_2 - x_1 & \ldots & y_{k-1} & x_k - x_{k-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{k-1} & y_{k-1} & \ldots & x_{2k-3} & y_{2k-3} \\ y_{k-1} & x_k - x_{k-1} & \ldots & y_{2k-3} & x_{2k-2} - x_{2k-3} \end{vmatrix} \tag{3.15}$$

and

$$\beta_{2k} = (-1)^{k} 2^{(2k-1)^2} \begin{vmatrix} x_0 & y_0 & \ldots & x_{k-1} & y_{k-1} \\ y_0 & x_1 - x_0 & \ldots & y_{k-1} & x_k - x_{k-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{k-1} & y_{k-1} & \ldots & x_{2k-2} & y_{2k-2} \\ y_{k-1} & x_k - x_{k-1} & \ldots & y_{2k-2} & x_{2k-1} - x_{2k-2} \end{vmatrix} \tag{3.16}$$

$$\beta_{2k+1} = (-1)^{k} 2^{(2k)^2} \begin{vmatrix} x_0 & y_0 & \ldots & x_{k-1} & y_{k-1} & x_k \\ y_0 & x_1 - x_0 & \ldots & y_{k-1} & x_k - x_{k-1} & y_k \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ x_{k-1} & y_{k-1} & \ldots & x_{2k-2} & y_{2k-2} & x_{2k-1} \\ y_{k-1} & x_k - x_{k-1} & \ldots & y_{2k-2} & x_{2k-1} - x_{2k-2} & y_{2k-1} \\ x_k & y_k & \ldots & x_{2k-1} & y_{2k-1} & x_{2k} \end{vmatrix}. \tag{3.17}$$

We can express α_N and β_N in terms of the Hankel determinants by taking the condition $r_j^2 = \rho^2 + (z + z_j)^2$ into consideration (see appendix C):

$$\alpha_N = (-1)^{\frac{1}{2}(N-1)(N+2)} \frac{2^{(N-1)^2}}{\rho^{(N-1)^2}} \begin{vmatrix} v_0 & v_1 & \ldots & v_{N-2} \\ v_1 & v_2 & \ldots & v_{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ v_{N-2} & v_{N-1} & \ldots & v_{2N-4} \end{vmatrix} \tag{3.18}$$

and

$$\beta_N = (-1)^{\frac{1}{2}N(N-1)} \frac{2^{(N-1)^2}}{\rho^{N(N-2)}} \begin{vmatrix} w_0 & w_1 & \ldots & w_{N-1} \\ w_1 & w_2 & \ldots & w_N \\ \vdots & \vdots & \ddots & \vdots \\ w_{N-1} & w_N & \ldots & w_{2N-2} \end{vmatrix} \tag{3.19}$$

where

$$v_n = \sum_{j=1}^{2N} C_j S_j z_j^n \qquad w_n = \sum_{j=1}^{2N} \frac{C_j}{S_j^*} z_j^n. \tag{3.20}$$

We have the relation between some Hankel determinants and double Casorati determinants:

$$\begin{vmatrix} t_0 & t_1 & \dots & t_{N-1} \\ t_1 & t_2 & \dots & t_N \\ \vdots & \vdots & \ddots & \vdots \\ t_{N-1} & t_N & \dots & t_{2N-2} \end{vmatrix} = (-1)^{\frac{1}{2}N(N-1)} Z_M^{-1} \begin{vmatrix} \phi_{11} & \dots & \phi_{M1} \\ \phi_{12} & \dots & \phi_{M2} \\ \vdots & \ddots & \vdots \\ \phi_{1N} & \dots & \phi_{MN} \\ \psi_{11} & \dots & \psi_{M1} \\ \psi_{12} & \dots & \psi_{M2} \\ \vdots & \ddots & \vdots \\ \psi_{1,M-N} & \dots & \psi_{M,M-N} \end{vmatrix} \tag{3.21}$$

where

$$\begin{aligned} t_n &= \sum_{j=1}^{M} z_j^n e_j \\ \phi_{jn} &= z_j^{n-1} \prod_{k=1,k\neq j}^{M} (z_k - z_j) e_j \qquad \psi_{jn} = z_j^{n-1}. \end{aligned} \tag{3.22}$$

By virtue of this relation we can link the SS solution with the NK solution:

$$\alpha_N = (-1)^{N+1} \frac{2^{(N-1)^2}}{\rho^{(N-1)^2}} Z_{2N}^{N-2} g_{2N} \tag{3.23}$$

$$\beta_N = \frac{2^{(N-1)^2}}{\rho^{N(N-2)}} \frac{Z_{2N}^{N-1} f_{2N}^*}{S_1^* S_2^* \dots S_{2N}^*} \tag{3.24}$$

$$\gamma_N = (-1)^N \frac{2^{N^2}}{\rho^{N^2}} Z_{2N}^{N-1} f_{2N} \tag{3.25}$$

$$\delta_N = \frac{2^{N^2}}{\rho^{N^2-1}} \frac{Z_{2N}^{N} g_{2N}^*}{S_1^* S_2^* \dots S_{2N}^*}. \tag{3.26}$$

Equations (3.23)–(3.26) and equations (3.1)–(3.4) immediately lead to equations (2.26)–(2.29) and, therefore, the SS conjecture has been proved explicitly.

4. The extension of Sasa–Satsuma solutions

In the direct method, the Ernst equation is decomposed to the bilinear forms in many different ways and correspondingly we obtain various series of solutions. Also the special decomposition of the Ernst equations (2.7)–(2.9) or equivalently (2.16)–(2.18) enable us to extend the SS solutions furthermore. Namely by the help of equations (2.7)–(2.9) and (2.16)–(2.18) we can generalize the functions F, G, H and K in equations (2.7)–(2.9) as

$$F = d\rho^{n_0} A^{(n)} \tag{4.1}$$

$$G = \rho^{n_0}[a_1 \tilde{A}^{(n+1)} + b_1 \rho^{n-1} A^{(n+1)} + c_1 \rho^{1-n} A^{(n-1)}] \tag{4.2}$$

$$H = \rho^{n_0}[a_2 \tilde{A}^{(n+1)} + b_2 \rho^{n-1} A^{(n+1)} + c_2 \rho^{1-n} A^{(n-1)}] \tag{4.3}$$

$$K = \rho^{n_0}[a_3 \tilde{A}^{(n+1)} + b_3 \rho^{n-1} A^{(n+1)} + c_3 \rho^{1-n} A^{(n-1)}] \tag{4.4}$$

with the constants $a_i, b_i, c_i (i = 1, 2, 3)$ and d. However F, G, H and K are not independent and subject to equation (2.6) or equivalently the Jacobi identity (2.15). Hence these constants must satisfy the relations:

$$a_1 b_3 + a_3 b_1 = 2a_2 b_2 \tag{4.5}$$

$$a_1c_3 + a_3c_1 = 2a_2c_2 \tag{4.6}$$
$$b_1b_3 = b_2^2 \tag{4.7}$$
$$c_1c_3 = c_2^2 \tag{4.8}$$
$$a_1a_3 - a_2^2 = d^2 \tag{4.9}$$
$$b_1c_3 + b_3c_1 - 2b_2c_2 = d^2. \tag{4.10}$$

Then we have two series of exact solutions of equations (2.1) and (2.2):

$$\tilde{f} = \frac{F}{G} \qquad \psi = \frac{H}{G} \tag{4.11}$$
$$\tilde{f} = \frac{F}{K} \qquad \psi = \frac{H}{K} \tag{4.12}$$

which includes both the SS solutions ($a_1 = a_3 = d = 1$ and $a_2 = 0$) and Nakamura's solutions ($c_1 = b_3 = d = 1, a_2 = i$ and others $= 0$) [7]. The relation between our solutions (4.11), (4.12) and the NK solutions on an arbitrary background, for instance, the extension to the Einstein–Maxwell system [11] or the Korotkin–Matveev solutions [12] is still not obvious.

5. Concluding remarks and discussion

In the previous section we analytically proved that the SS solutions include the NK solutions in a particular case. Substituting equations (2.26)–(2.29) into equations (2.16)–(2.18) we have the bilinear forms [9] satisfied by NK solutions:

$$\left[D_\rho^2 + \frac{1}{\rho}D_\rho + D_z^2\right](f_{2N}g_{2N}^*)\cdot(g_{2N}g_{2N}^* - f_{2N}f_{2N}^*) = 0 \tag{5.1}$$
$$\left[D_\rho^2 + \frac{1}{\rho}D_\rho + D_z^2\right](f_{2N}^*g_{2N})\cdot(g_{2N}g_{2N}^* - f_{2N}f_{2N}^*) = 0 \tag{5.2}$$
$$\left[D_\rho^2 + \frac{1}{\rho}D_\rho + D_z^2\right](g_{2N}g_{2N}^* + f_{2N}f_{2N}^*)\cdot(g_{2N}g_{2N}^* - f_{2N}f_{2N}^*) = 0. \tag{5.3}$$

We have indirectly proved equations (5.1)–(5.3) through the Bäcklund transformation. However, it is another mathematically interesting problem to prove equations (5.1)–(5.3) directly. This, however, is still an open question. In [1, 2], we were confronted with the same problem in Tomimatsu–Sato solutions, that is, direct proof of Nakamura's conjecture which will be explained briefly. Before discussing this problem we add one more note in relation with equations (5.1)–(5.3). Namely there is another series of exact solutions called the extended-TS solutions [13], $\xi_n = g'_n/f'_n$. The bilinear forms satisfied by them have been also proposed [9]:

$$L(f'_n g'^*_n)\cdot(g'_n g'^*_n - f'_n f'^*_n) = 0 \tag{5.4}$$
$$L(f'^*_n g'_n)\cdot(g'_n g'^*_n - f'_n f'^*_n) = 0 \tag{5.5}$$
$$L(g'_n g'^*_n + f'_n f'^*_n)\cdot(g'_n g'^*_n - f'_n f'^*_n) = 0 \tag{5.6}$$

where

$$L = (x^2 - 1)D_x^2 + x(D_x + \partial_x) - (y^2 - 1)D_y^2 - y(D_y + \partial_y). \tag{5.7}$$

The independent variables x and y are usual prolated spheroidal coordinates which are connected to ρ and z by

$$\rho = K(x^2 - 1)^{1/2}(1 - y^2)^{1/2} \qquad z = Kxy + \zeta \tag{5.8}$$

with constants K and ζ. To obtain the explicit expressions of the extended-TS solutions and to prove equations (5.4)–(5.6) for arbitrary n are unsolved problems.

Let us return to the problem of the TS solution. A Nakamura gave the following conjecture (Nakamura's conjecture) [14]: The general solution of Toda molecule with n lattice cites reproduces the TS solution with the deformation parameter $\delta = n$ in a particular choice of initial function. Direct proof of this conjecture is successful only in the restricted case, Weyl solution, which is obtained by the dimensional reduction by one. The trouble comes from the fact that the TS solution is embedded in two-dimensional space. Concretely speaking, the reason for the difficulties comes from the fact that two directional Wronskians appeared in two dimensions prohibits Plücker's identity, one of Pfaffian's identities, unlike in the one-dimensional case. We also discussed the same problem from a quite different approach, acceleration method [2]. However, the situation is quite the same as in the direct method. In this case the trouble is that the addition theorem [15], the key formula for the proof, is only valid in one dimension. Thus, for the sake of completeness we may be forced to go beyond the bilinear formalism or to develop a new acceleration method. The proof of the SS conjecture in this paper has circumvented this trouble in the two-dimensional case by the help of the Bäcklund transformation. So this may be a useful hint to the extensions mentioned above.

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Appendix A

First, we show equations (3.14) and (3.15). Defining E_k and O_k as

$$E_k \equiv u_{2k-2} + u_{2k} \tag{A1}$$

$$O_k \equiv u_{2k-1} + u_{2k+1} \qquad (k = 1, 2, 3, \ldots) \tag{A2}$$

it follows from equations (3.9)–(3.11) that

$$\alpha_N = \begin{vmatrix} E_1 & -O_1 & E_2 & -O_2 & \cdots \\ O_1 & E_1 - E_2 & -O_1 + O_2 & E_2 - E_3 & \cdots \\ E_2 & O_1 - O_2 & E_1 - E_2 + E_3 & -O_1 + O_2 - O_3 & \cdots \\ O_2 & E_2 - E_3 & O_1 - O_2 + O_3 & E_1 - E_2 + E_3 - E_4 & \cdots \\ E_3 & O_2 - O_3 & E_2 - E_3 + E_4 & O_1 - O_2 + O_3 - O_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}. \tag{A3}$$

In the case of the special solutions (3.9)–(3.11) we have

$$E_k = E_k^{(0)} \qquad O_k = O_k^{(0)} \tag{A4}$$

where we have introduced $E_k^{(a)}$ and $O_k^{(a)}$ as

$$E_k^{(a)} = \sum_{n=1}^{k} (-1)^{k-n} \frac{2k-1}{2n-1} {}_{k-2+n}C_{k-n} 2^{2n-1} x_{n+a} \tag{A5}$$

$$O_k^{(a)} = \sum_{n=1}^{k} (-1)^{k+1-n} {}_{k-1+n}C_{k-n} 2^{2n} y_{n+a}. \tag{A6}$$

We substitute equations (A5) and (A6) into equation (A3) and transform equation (A3) according to the following procedures in the 'step n':

(i) $((2m+1)$th row$)$ $-(-1)^{m+1-n}\frac{2m+1}{2n-1}\,{}_{m-1+n}C_{m+1-n}\times$ $((2n-1)$th row$)$, $m=n,n+1,\ldots,[\frac{N}{2}]-1$.

(ii) The similar calculations for columns.

(iii) $((2m+2)$th row$)$ $-(-1)^{m+1-n}\,{}_{m+n}C_{m+1-n}\times$ $((2n)$th row$)$, $m=n,n+1,\ldots,[\frac{N-1}{2}]-1$.

(iv) The similar calculations for columns.

Calculation from the step 1 to m for α_N leads to equations (3.14) and (3.15), which we will show by induction.

Suppose that the above statement is correct for $N=1,2,\ldots 2k$. Application of the procedures for α_{2k} to α_{2k+1} leads to the expression as

$$\alpha_{2k+1}=\begin{vmatrix} 2x_1 & 4y_1 & 8x_2 & 16y_2 & \ldots & 2^{2k-1}x_k & -c_{2k,1} \\ -4y_1 & 8(x_1-x_2) & -16y_2 & 32(x_2-x_3) & \ldots & -2^{2k}y_k & c_{2k,2} \\ 8x_2 & 16y_2 & 32x_3 & 64y_3 & \ldots & 2^{2k+1}x_{k+1} & -c_{2k,3} \\ -16y_2 & 32(x_2-x_3) & -64y_3 & 128(x_3-x_4) & \ldots & -2^{2k+2}y_{k+1} & c_{2k,4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2^{2k-1}x_k & 2^{2k}y_k & 2^{2k+1}x_{k+1} & 2^{2k+2}y_{k+1} & \ldots & 2^{4k-3}x_{2k-1} & -c_{2k,2k-1} \\ c_{2k,1} & c_{2k,2} & c_{2k,3} & c_{2k,4} & \ldots & c_{2k,2k-1} & c_{2k,2k} \end{vmatrix} \tag{A7}$$

with

$$c_{2k,2n+1}=\sum_{m=0}^{2n}{}_{2n}C_m O^{(0)}_{k-n+m} \qquad (n=0,1,\ldots,k-1) \tag{A8}$$

$$c_{2k,2n+2}=\sum_{m=0}^{2n}{}_{2n}C_m (E^{(0)}_{k-n+m}-E^{(0)}_{k-n+m+1}) \qquad (n=0,1,\ldots,k-2) \tag{A9}$$

$$c_{2k,2k}=\sum_{m=1}^{2k}(-1)^{m-1}E^{(0)}_m. \tag{A10}$$

Equations (A8) and (A9) were derived from the relations:

$$\sum_{m=0}^{2l}(-1)^m O^{(0)}_{k-l+m}-\sum_{n=0}^{l-1}(-1)^{l-n}\frac{2l+1}{2n+1}\,{}_{l+n}C_{l-n}\sum_{m=0}^{2n}{}_{2n}C_m O^{(0)}_{k-n+m}=\sum_{m=0}^{2l}{}_{2l}C_m O^{(0)}_{k-l+m} \tag{A11}$$

$$\sum_{m=0}^{2l+1}(-1)^m E^{(0)}_{k-l+m}-\sum_{n=0}^{l-1}(-1)^{l-n}\,{}_{l+n+1}C_{l-n}\sum_{m=0}^{2n}{}_{2n}C_m (E^{(0)}_{k-n+m}-E^{(0)}_{k-n+m+1})$$
$$=\sum_{m=0}^{2l}{}_{2l}C_m (E^{(0)}_{k-l+m}-E^{(0)}_{k-l+m+1}). \tag{A12}$$

For detail see appendix B.

It is easily seen that

$$\sum_{m=0}^{2n}{}_{2n}C_m O^{(0)}_{k-n+m}=2^{2n}O^{(n)}_k \tag{A13}$$

$$\sum_{m=0}^{2n}{}_{2n}C_m (E^{(0)}_{k-n+m}-E^{(0)}_{k-n+m+1})=2^{2n}(E^{(n)}_k-E^{(n)}_{k+1}). \tag{A14}$$

Performing the procedures from the step 1 to step m in equation (A3), we have

$$\alpha_{2k+1} = \begin{vmatrix} 2x_1 & 4y_1 & 8x_2 & 16y_2 & \dots & 2^{2k-1}x_k & -O_k^{(0)} \\ -4y_1 & 8(x_1-x_2) & -16y_2 & 32(x_2-x_3) & \dots & -2^{2k}y_k & E_k^{(0)}-E_{k+1}^{(0)} \\ 8x_2 & 16y_2 & 32x_3 & 64y_3 & \dots & 2^{2k+1}x_{k+1} & -2^2 O_k^{(1)} \\ -16y_2 & 32(x_2-x_3) & -64y_3 & 128(x_3-x_4) & \dots & -2^{2k+2}y_{k+1} & 2^2(E_k^{(1)}-E_{k+1}^{(1)}) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2^{2k-1}x_k & 2^{2k}y_k & 2^{2k+1}x_{k+1} & 2^{2k+2}y_{k+1} & \dots & 2^{4k-3}x_{2k-1} & -2^{2(k-1)}O_k^{(k-1)} \\ O_k^{(0)} & E_k^{(0)}-E_{k+1}^{(0)} & 2^2 O_k^{(1)} & 2^2(E_k^{(1)}-E_{k+1}^{(1)}) & \dots & 2^{2(k-1)}O_k^{(k-1)} & c_{2k,2k} \end{vmatrix}. \tag{A15}$$

The procedures,

(i) $(2k)$th row $-(-1)^{k-n}\,{}_{k-1+n}C_{k-n}\times$ $(2n)$th row, $n = 1, 2, \dots, k-1$

(ii) the similar processes for columns,

reduce equation (A15) to

$$\alpha_{2k+1} = \begin{vmatrix} 2x_1 & 4y_1 & \dots & 2^{2k-1}x_k & 2^{2k}y_k \\ -4y_1 & 8(x_1-x_2) & \dots & -2^{2k}y_k & 2^{2k+1}(x_k-x_{k+1}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2^{2k-1}x_k & 2^{2k}y_k & \dots & 2^{4k-3}x_{2k-1} & 2^{4k-2}y_{2k-1} \\ -2^{2k}y_k & 2^{2k+1}(x_k-x_{k+1}) & \dots & -2^{4k-2}y_{2k-1} & 2^{4k-1}(x_{2k-1}-x_{2k}) \end{vmatrix}. \tag{A16}$$

Likewise, α_{2k+2} is expressed as

$$\alpha_{2k+2} = \begin{vmatrix} 2x_1 & 4y_1 & \dots & 2^{2k-1}x_k & 2^{2k}y_k & c_{2k+1,1} \\ -4y_1 & 8(x_1-x_2) & \dots & -2^{2k}y_k & 2^{2k+1}(x_k-x_{k+1}) & -c_{2k+1,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2^{2k-1}x_k & 2^{2k}y_k & \dots & 2^{4k-3}x_{2k-1} & 2^{4k-2}y_{2k-1} & c_{2k+1,2k-1} \\ -2^{2k}y_k & 2^{2k+1}(x_k-x_{k+1}) & \dots & -2^{4k-2}y_{2k-1} & 2^{4k-1}(x_{2k-1}-x_{2k}) & -c_{2k+1,2k} \\ c_{2k+1,1} & c_{2k+1,2} & \dots & c_{2k+1,2k-1} & c_{2k+1,2k} & c_{2k+1,2k+1} \end{vmatrix} \tag{A17}$$

with

$$c_{2k+1,2n+1} = \sum_{m=0}^{2n} {}_{2n}C_m E_{k+1-n+m}^{(0)} \tag{A18}$$

$$c_{2k+1,2n+2} = \sum_{m=0}^{2n} {}_{2n}C_m (O_{k-n+m}^{(0)} - O_{k+1-n+m}^{(0)}) \qquad (n = 0, 1, \dots, k-1) \tag{A19}$$

$$c_{2k+1,2k+1} = \sum_{m=1}^{2k+1} (-1)^{m-1} E_m^{(0)}. \tag{A20}$$

It is easily checked that

$$\sum_{m=0}^{2n} {}_{2n}C_m E_{k+1-n+m}^{(0)} = 2^{2n} E_{k+1}^{(n)} \tag{A21}$$

$$\sum_{m=0}^{2n} {}_{2n}C_m (O_{k-n+m}^{(0)} - O_{k+1-n+m}^{(0)}) = 2^{2n+1} \tilde{E}_{k+1}^{(n)} \tag{A22}$$

with

$$\tilde{E}_k^{(a)} = \sum_{n=1}^{k} (-1)^{k-n} \frac{2k-1}{2n-1}\, {}_{k-2+n}C_{k-n} 2^{2n-1} y_{n+a}. \tag{A23}$$

Then we have

$$\alpha_{2k+2} = \begin{vmatrix} 2x_1 & 4y_1 & \dots & 2^{2k-1}x_k & 2^{2k}y_k & E_{k+1}^{(0)} \\ -4y_1 & 8(x_1-x_2) & \dots & -2^{2k}y_k & 2^{2k+1}(x_k-x_{k+1}) & -2\tilde{E}_{k+1}^{(0)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2^{2k-1}x_k & 2^{2k}y_k & \dots & 2^{4k-3}x_{2k-1} & 2^{4k-2}y_{2k-1} & 2^{2k-2}E_{k+1}^{(k-1)} \\ -2^{2k}y_k & 2^{2k+1}(x_k-x_{k+1}) & \dots & -2^{4k-2}y_{2k-1} & 2^{4k-1}(x_{2k-1}-x_{2k}) & -2^{2k-1}\tilde{E}_{k+1}^{(k-1)} \\ E_{k+1}^{(0)} & 2\tilde{E}_{k+1}^{(0)} & \dots & 2^{2k-2}E_{k+1}^{(k-1)} & 2^{2k-1}\tilde{E}_{k+1}^{(k-1)} & c_{2k+1,2k+1} \end{vmatrix}. \tag{A24}$$

The procedures,

(i) $(2k+1)$th row $-(-1)^{k+1-n}\frac{2k+1}{2n-1}\,{}_{k-1+n}C_{k+1-n}\times$ $(2n-1)$th row, $n = 0, 1, \dots, k$

(ii) the similar calculations for columns,

give

$$\alpha_{2k+2} = \begin{vmatrix} 2x_1 & 4y_1 & \dots & 2^{2k-1}x_k & 2^{2k}y_k & 2^{2k+1}x_{k+1} \\ -4y_1 & 8(x_1-x_2) & \dots & -2^{2k}y_k & 2^{2k+1}(x_k-x_{k+1}) & -2^{2k+2}y_{k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2^{2k-1}x_k & 2^{2k}y_k & \dots & 2^{4k-3}x_{2k-1} & 2^{4k-2}y_{2k-1} & 2^{4k-1}x_{2k} \\ -2^{2k}y_k & 2^{2k+1}(x_k-x_{k+1}) & \dots & -2^{4k-2}y_{2k-1} & 2^{4k-1}(x_{2k-1}-x_{2k}) & -2^{4k}y_{2k} \\ 2^{2k+1}x_{k+1} & 2^{2k+2}y_{k+1} & \dots & 2^{4k-1}x_{2k} & 2^{4k}y_{2k} & 2^{4k+1}x_{2k+1} \end{vmatrix}. \tag{A25}$$

Therefore we have verified equations (3.14) and (3.15).

Next, we will show equations (3.16) and (3.17). By means of E_k and O_k, β_N is expressed as

$$\beta_N = \begin{vmatrix} u_0 & -u_1 & u_2 & -u_3 & u_4 & \dots \\ 2u_1 & 2u_0-E_1 & -2u_1+O_1 & 2u_2-E_2 & -2u_3+O_2 & \dots \\ 2u_2 & 2u_1-O_1 & 2u_0-(E_1-E_2) & -2u_1+(O_1-O_2) & 2u_2-(E_2-E_3) & \dots \\ 2u_3 & 2u_2-E_2 & 2u_1-(O_1-O_2) & 2u_0-(E_1-E_2+E_3) & -2u_1+(O_1-O_2+O_3) & \dots \\ 2u_4 & 2u_3-O_2 & 2u_2-(E_2-E_3) & 2u_1-(O_1-O_2+O_3) & 2u_0-(E_1-E_2+E_3-E_4) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}. \tag{A26}$$

Let us transform equation (A26) along the procedures:

(i) Third row $+(2\times 1)$th row,

(ii) mth row $+$ $(m-2)$th row, $m = 4, 5, \dots, N$,

(iii) mth column $+$ $(m-2)$th column, $m = 3, 4, \dots, N$.

Substituting (3.9)–(3.11) into it, we obtain

$$\beta_N = \begin{vmatrix} x_0 & y_0 & E_1^{(0)} & -O_1^{(0)} & E_2^{(0)} & \dots \\ -2y_0 & 2(x_0-x_1) & O_1^{(0)} & E_1^{(0)}-E_2^{(0)} & -O_1^{(0)}+O_2^{(0)} & \dots \\ 2E_1^{(0)} & -O_1^{(0)} & 2^2E_1^{(1)} & -2^2O_1^{(1)} & 2^2E_2^{(1)} & \dots \\ 2O_1^{(0)} & E_1^{(0)}-E_2^{(0)} & 2^2O_1^{(1)} & 2^2(E_1^{(1)}-E_2^{(1)}) & -2^2(O_1^{(1)}-O_2^{(1)}) & \dots \\ 2E_2^{(0)} & O_1^{(0)}-O_2^{(0)} & 2^2E_2^{(1)} & 2^2(O_1^{(1)}-O_2^{(1)}) & 2^2(E_1^{(1)}-E_2^{(1)}+E_3^{(1)}) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}. \tag{A27}$$

The similar procedures for α_N lead us to equations (3.16) and (3.17).

Appendix B

Here we show equations (A11) and (A12). From the comparison of the coefficients of both hand sides of equations (A11) and (A12), it follows that they are equivalent to the relations for $a = 0, 1, \dots, n$:

$$\sum_{k=a}^{n}(-1)^{k-a}\,{}_{n+k+1}C_{n-k}\,{}_{2k}C_{k-a} = \begin{cases} 1 & n-a\text{: even} \\ 0 & n-a\text{: odd.} \end{cases} \tag{B1}$$

It should be noticed that

$$\sum_{k=0}^{n}(-1)^k \frac{(n+k+1)!}{(k!)^2(n-k)!}x^{k+n} = \frac{\mathrm{d}}{\mathrm{d}x}[x^{n+1}P_n(1-2x)] \tag{B2}$$

where $P_n(1-2x)$ are the Legendre polynomials.

B.1. The case $a = 0$

From equation (B2), we have

$$\sum_{k=0}^{n}(-1)^k \, {}_{n+k+1}C_{n-k} \, {}_{2k}C_k = \tfrac{1}{2}\int_0^1 \mathrm{d}x \, x^{-n-\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}x}[x^{n+1}P_n(1-2x)] = \tfrac{1}{2}[1+(-1)^n]. \tag{B3}$$

B.2. The case $a \geqslant 1$

We have

$$\sum_{k=a}^{n}(-1)^{k-a} \, {}_{n+k+1}C_{n-k} \, {}_{2k}C_{k-a} = \sum_{p=0}^{a}(-1)^{p-a} \, {}_aC_p \sum_{k=0}^{n}(-1)^k \, {}_{n+k+1}C_{n-k} \, {}_{2k+a-p}C_k. \tag{B4}$$

From equation (B2), we obtain for $p = 0, 1, \ldots, a-1$,

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k \, {}_{n+k+1}C_{n-k} \, {}_{2k+a-p}C_k &= \sum_{k=0}^{n}(-1)^k \frac{(n+k+1)!}{(k!)^2(n-k)!}\sum_{j=1}^{a-p}\frac{q_j}{k+j} \\ &= \sum_{j=1}^{a-p} q_j \int_0^1 \mathrm{d}x \, x^{-n+j-1}\frac{\mathrm{d}}{\mathrm{d}x}[x^{n+1}P_n(1-2x)] = 2^{a-p-1}(-1)^n \end{aligned} \tag{B5}$$

with $\sum_{j=1}^{a-p} q_j = 2^{a-p-1}$. Thus we can verify equation (B1) as follows

$$\sum_{k=a}^{n}(-1)^{k-a} \, {}_{n+k+1}C_{n-k} \, {}_{2k}C_{k-a} = \tfrac{1}{2} + \sum_{p=0}^{a}(-1)^{p-a} \, {}_aC_p 2^{a-p-1}(-1)^n = \tfrac{1}{2}[1+(-1)^{n-a}]. \tag{B6}$$

Appendix C

Here we show equation (3.19). Let us define X_n and Y_n as

$$\begin{aligned} X_n &\equiv \sum_{j=1}^{2N} C_j \mathrm{e}^{\mathrm{i}\omega_j} r_j^{2n-1} \\ Y_n &\equiv \sum_{j=1}^{2N} C_j \mathrm{e}^{\mathrm{i}\omega_j}(z+z_j) r_j^{2n-1}. \end{aligned} \tag{C1}$$

Then equations (3.16) and (3.17) are rewritten as

$$\beta_{2k} = (-1)^k \frac{2^{(2k-1)^2}}{\rho^{(2k)(2k-2)}} \begin{vmatrix} X_0 & Y_0 & \ldots & X_{k-1} & Y_{k-1} \\ Y_0 & X_1 - \rho^2 X_0 & \ldots & Y_{k-1} & X_k - \rho^2 X_{k-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ X_{k-1} & Y_{k-1} & \ldots & X_{2k-2} & Y_{2k-2} \\ Y_{k-1} & X_k - \rho^2 X_{k-1} & \ldots & Y_{2k-2} & X_{2k-1} - \rho^2 X_{2k-2} \end{vmatrix} \tag{C2}$$

$$\beta_{2k+1} = (-1)^k \frac{2^{(2k)^2}}{\rho^{(2k+1)(2k-1)}}$$

$$\times \begin{vmatrix} X_0 & Y_0 & \dots & X_{k-1} & Y_{k-1} & X_k \\ Y_0 & X_1 - \rho^2 X_0 & \dots & Y_{k-1} & X_k - \rho^2 X_{k-1} & Y_k \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ X_{k-1} & Y_{k-1} & \dots & X_{2k-2} & Y_{2k-2} & X_{2k-1} \\ Y_{k-1} & X_k - \rho^2 X_{k-1} & \dots & Y_{2k-2} & X_{2k-1} - \rho^2 X_{2k-2} & Y_{2k-1} \\ X_k & Y_k & \dots & X_{2k-1} & Y_{2k-1} & X_{2k} \end{vmatrix}. \tag{C3}$$

Introducing $p_m^{(k)}$ as

$$p_m^{(k)} = \sum_{j=1}^{2N} \frac{C_j}{S_j^*} z_j^k (bz_j + z_j^2)^m \tag{C4}$$

X_n and Y_n are expressed as

$$\begin{aligned} X_n &= \sum_{m=0}^{n} {}_nC_m a^{n-m} p_m^{(0)} \\ Y_n &= z \sum_{m=0}^{n} {}_nC_m a^{n-m} p_m^{(0)} + \sum_{m=0}^{n} {}_nC_m a^{n-m} p_m^{(1)} \end{aligned} \tag{C5}$$

where $a = \rho^2 + z^2$, $b = 2z$.

We will perform the following transformation in equations (C2) and (C3). First, the procedures,

(i) $(2n)$th row $-z\times$ $(2n-1)$th row,

(ii) the similar calculations for columns,

lead these equations to

$$\begin{vmatrix} \ddots & \vdots & \vdots & \\ \dots & X_n & Y_n & \dots \\ \dots & Y_n & X_{n+1} - \rho^2 X_n & \dots \\ & \vdots & \vdots & \ddots \end{vmatrix}$$

$$= \begin{vmatrix} \ddots & \vdots & \vdots & \\ \dots & \sum_{m=0}^{n} {}_nC_m a^{n-m} p_m^{(0)} & \sum_{m=0}^{n} {}_nC_m a^{n-m} p_m^{(1)} & \dots \\ \dots & \sum_{m=0}^{n} {}_nC_m a^{n-m} p_m^{(1)} & \sum_{m=0}^{n} {}_nC_m a^{n-m} p_m^{(2)} & \dots \\ & \vdots & \vdots & \ddots \end{vmatrix}. \tag{C6}$$

Second, repeating the procedures,

(i) $(2n)$th row $-a\times$ $(2n-2)$th row,

(ii) $(2n-1)$th row $-a\times$ $(2n-3)$th row,

we obtain

$$\text{(C6)} = \begin{vmatrix} \ddots & \vdots & \vdots & \\ \dots & p_n^{(0)} & p_n^{(1)} & \dots \\ \dots & p_n^{(1)} & p_n^{(2)} & \dots \\ & \vdots & \vdots & \ddots \end{vmatrix} \tag{C7}$$

where, from equations (3.20), $p_n^{(k)}$ is expressed in terms of w_n as

$$p_n^{(k)} = \sum_{m=0}^{n} {}_nC_m b^{n-m} w_{n+k+m}. \tag{C8}$$

Final procedures,

(i) $(2n-1)$th row $-b\times$ $(2n-2)$th row,

(ii) $(2n)$th row $-b\times$ $(2n-1)$th row,
show

$$
(\text{C7}) = \begin{vmatrix} \ddots & \vdots & \vdots & \\ \cdots & w_{2n} & w_{2n+1} & \cdots \\ \cdots & w_{2n+1} & w_{2n+2} & \cdots \\ & \vdots & \vdots & \ddots \end{vmatrix} \tag{C9}
$$

due to

$$
p_0^{(k+2n)} = w_{k+2n}. \tag{C10}
$$

Therefore, we have verified equation (3.19). Equation (3.18) is verified similarly.

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